Lecture 12 – Holomorphic Dynamics

Thm For a rational map f of degree ≥ 2, the number of attracting or neutral periodic cycles is $\leq 6d-6$.

(**Rmk**: Fatou-Shishikura : $\leq 2d-2$)

Prop.: Every basin of attracting or parabolic cycle contains a critical point.

Cor.: the number of attracting or parabolic cycles is $\leq 2d-2$.

$\boxed{Q}$ What about the other neutral cycles?

$$f^p(z_0) = z_0$$

$$(f^p)'(z_0) = e^{2\pi i \alpha}$$

$$\alpha \in \mathbb{R} \setminus \mathbb{Q}.$$

Lemma For a rational map of degree $d \geq 2$, the number of neutral cycles of $\lambda \neq 1$ is at most $4d-4$.

Pf.: Suppose there is $f(z)$ of $\deg(f) = d$ with at least $4d-3$ neutral cycles.

Define $\quad f_t(z) = \dfrac{(1-t)\,p(z) + t\,z^d}{(1-t)\,q(z) + t}$

where $f(z) = \dfrac{p(z)}{q(z)} = f_0(z)$

$f_1(z) = z^d$ has no neutral cycles

$f_1^n(z) = z^{d^n} = z \to z = 0$

$\to |z| = 1$

$f_1'(z) = d\,z^{d-1}$

Since f_0 has at least $4d-3$ periodic cycles of multiplier $\neq 1$, we can analytically extend the multipliers to holo

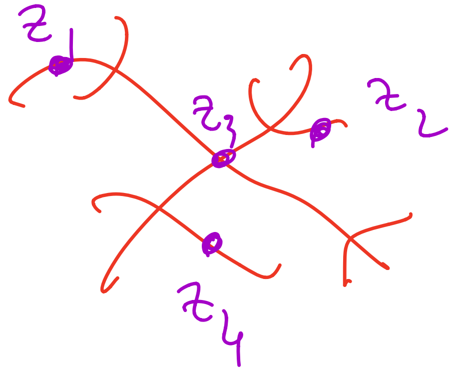
$t \mapsto (\lambda_1(t), \lambda_2(t), \dots, \lambda_j(t), \dots, \lambda_{4d-3}(t))$

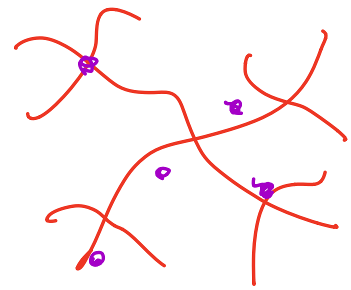
[By implicit function theorem:

$F(t,z) = (f_t(z))^p - z = 0$

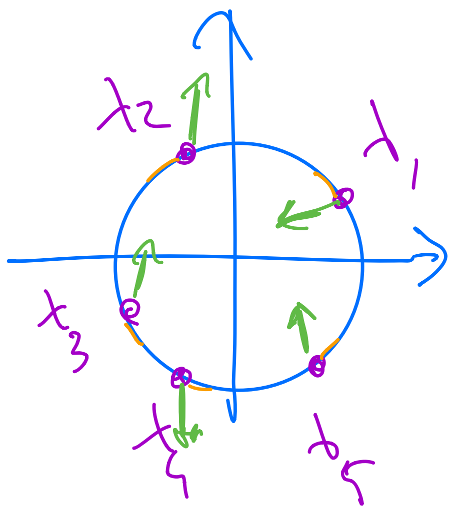
$\dfrac{\partial F}{\partial z} = (f_t(z)^p)' - 1 \neq 0$

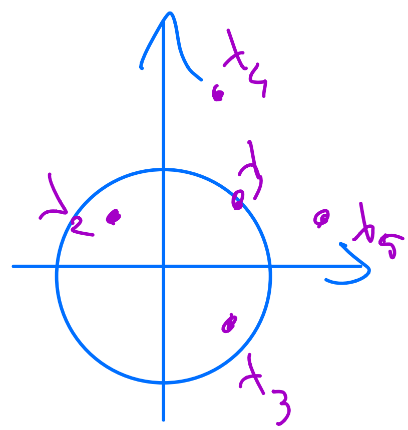
hence you can write $z_1(t), \dots, z_{4d-3}(t)$ the continuations of the periodic cycles]





$f_0(z)$ $f_t(z)$





$4d-3$

Goal: show that at least half of them goes inside
$(2d-1)$

$\Downarrow$

CONTRADICTION

Lemma

$t \mapsto \lambda_j(t)$ is not constant on $t \in [0,\varepsilon)$

Pf If so, then $\lambda_j(t)$ can be analytically continued to be constant for $t \in [0,1]$. But $f_1(z) = z^d$ has <u>no</u> neutral cycles.

We can write for each j

$$\frac{\lambda_j(t)}{\lambda_j(0)} = 1 + a_j t^{n(j)} + \text{h.o.t.}$$

$$\log \left| \frac{\lambda_j(t)}{\lambda_j(0)} \right| = \log |\lambda_j(t)| =$$

$$= \mathrm{Re}\left(\log\left(1 + a_j t^{n(j)} + \dots\right)\right)$$

$t = re^{i\theta}$

$$\int \log |\lambda_j(re^{i\theta})| \, d\theta = 0$$

$$\int \sum_{j=1}^{4d-3} \log |\lambda_j(re^{i\theta})| \, d\theta = 0$$

$$\mathrm{sign}(\log|\lambda|) = \begin{cases} +1 & \text{if } |\lambda| > 1 \\ -1 & \text{if } |\lambda| < 1 \end{cases}$$

Lemma If, for some r small,

$$\sigma_j(\theta) := \mathrm{sign}\left(\log|\lambda_j(re^{i\theta})|\right) \in \{0, \pm 1\}$$

Then $\displaystyle\int \sum_{j=1}^{4d-3} \sigma_j(\theta)\, d\theta = 0$.

Hence, there is θ s.t.

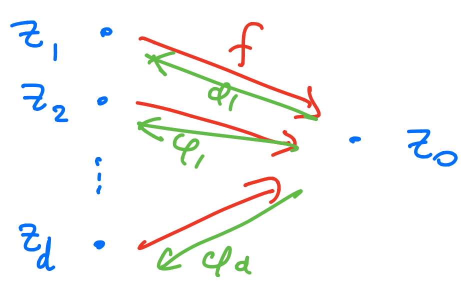
$$\sum_{j=1}^{4d-3} \sigma_j(\theta) \leq -1$$

But this implies that there are at least $2d-1$ values of j for which $|\lambda_j(re^{i\theta})| < 1$.

Hence, $f_{re^{i\theta}}(z)$ has at least $2d-1$ attracting cycles. This contradicts the previous Lemma.

Thm The Julia set of a rational map of degree ≥ 2 is the closure of its set of repelling periodic points.

Pf (Fatou)
Let $z_0 \in J(f)$ which is not a fixed pt or a critical value.



Let $\varphi_1(z), \dots, \varphi_d(z)$ the local inverses of f. $(z_j = \varphi_j(z_0).)$

$$g_n(z) := \frac{(f^n(z) - \varphi_1(z))(z - \varphi_2(z))}{(f^n(z) - \varphi_2(z))(z - \varphi_1(z))}$$

<u>Claim</u> For every $N \ni z_0$, $\exists\, z \in N$, $n > 0$

s.t. $$f^n(z) = \begin{cases} z & \longrightarrow z = f^n(z) \\ \varphi_1(z) & \longrightarrow z = f^{n+1}(z) \\ \varphi_2(z) & \nearrow \end{cases}$$

If not, $(g_n|_N(z)) \subseteq \hat{\mathbb{C}} \setminus \{0, 1, \infty\}$

<u>Hence</u> by Montel, $(g_n|_N)$ is normal

<u>hence</u> also $(f^n|_N)$ is normal.

But this contradicts $z_0 \in J(f)$.

Hence every nbd of z_0 contains a periodic cycle, and there are only finitely many which are not repelling.